Computer Generated Time-Tables and Bus Schedules for a Large Bus Transport Network

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This report describes a method developed for analyzing a large bus transport on a digital computer with a view to improving the utilization of buses and the duty allocation for the crew without making elaborate structural changes in the existing setup. With the a priori statistics concerning the alighting and boarding of passengers at each stage of a route for different times of the day and the a posteriori statistics concerning the waiting time of the passengers at various stages of the route as the information supplied by the transport organization, a computer program has been developed to realize time-tables and bus schedules that can optimize the fixed plus variable cost of operation subject to the requirements and constraints imposed on the optimization procedure by the passengers, the crew, and the management.

Large bus transport networks frequently come across problems of planning based upon the optimization of one or more objective functions that are too complex to be analyzed manually. Such problems naturally fall in the domain of operations research that can be handled efficiently through the use of digital computers.

A fundamental motivation of operations research relative to a transport undertaking is a thorough investigation of its objectives in relation to the requirements of and constraints imposed by the passengers, the crew, and the management. Whereas the requirements broadly define the measures that ought to be taken in realizing a high enough efficiency with which the defined objectives are realized, the constraints impose conditions on the factors that influence the achievement of the stated objectives. Thus, the problem is one of finding an optimum solution to a situation in which the requirements and constraints have opposing influence on the stated objectives. Each specifies an upper and a lower bound for the influence of their effect upon the setup for a proper functioning of the transport organization.

A natural approach to the foregoing problem will be to find a solution that minimizes the total operational cost, viz., the sum of the fixed cost and the variable cost, with the parameter values lying between the lower and upper bounds prescribed by the conditions of the problem. In a profit-motivated undertaking this would amount to maximizing the profit and in a nonprofit organization, to a minimization of the operational cost. However, both are equivalent as far as the mathematical formulation of the problem is concerned.

A computer-based planning of a transport network involves the following five major stages of investigation:

1. Given the data obtained by a commuter survey, one is required to realize an optimum transport network.
2. Given the statistics about the number of passengers alighting and boarding at each stage (or stop) for each bus on a route and the waiting time of the passengers at different times of the day at each stage, one is required to develop an optimum headway list.
3. Given the optimum headway list, one must plan an optimum bus schedule that can readily be implemented.
4. Given the optimum time-table, one must assign the crews to the buses in such a manner that maximum work is realized from them (with due regard to the labor law).
5. Given the time-table, the bus schedule, and the crew schedule one must realize an optimum switching schedule for buses operating on routes within each subnetwork or compact area.

ELIAS,[2, 3] LINES, LAMPKIN, AND SAALMANS,[4] AND BENNETT AND POTTS[1] directed their investigations to realizing economic scheduling for both man and machine in public transportation, to designing routes and service frequencies for a municipal bus company, and to a rostering problem in transportation, respectively. However, their methods do not take into consideration a limitation imposed on the bus-transport organization in a developing country, viz., the nonavailability in transport organizations of an adequate number of competent people to collect sophisticated statistics to a reasonable degree of accuracy. While PULLEN AND WEBB[5] have given a scheme for the preparation of the van driver's duty schedule for the bulk conveyance of mail in the inner area of the London Postal region of the British Post Office, their method is unsuitable for passenger transport. YOUNGER AND WILKINSON[7] gave a technique for scheduling buses and the efficient utilization of available part-time crews. They mentioned that the economy effected was not commensurate with the cost of computation.

This paper describes a solution to the second and third problems given above, subject to the constraint that only unskilled labor is available for data collection. Consequently, data collection should be reasonably straightforward, if reliable data is to be obtained.

The program for making a timetable and a bus schedule is written only with a view to decreasing the operational cost of the undertaking. It should be made clear that the object of the investigation is not to increase the convenience of the passengers further. Thus, the investigation centers on the development of methods that will decrease the variable cost and the fixed cost incurred by the undertaking without affecting passenger convenience. Consequently, the change in the timetable and the bus schedule will hardly be noticed by passengers, although an economy will be realized by management.

A main factor affecting passengers' convenience is maximum waiting time. The method that is developed in what follows, effects a uniformity in waiting-time distribution with maximum waiting time corresponding to that in the existing setup.

The next section gives the organizational structure of the network under investigation. The analysis being strongly dependent upon the accuracy and type of statistics collected, the human factors involved are analyzed in the third section together with the method of data collection for realizing a reasonable degree of reliability. In deciding the criteria and the methods of analysis, the requirements called for, and the constraints imposed by the passengers, the crew, and the management all have to be considered. These are described in detail, in the fourth section. The convenience of the passengers being a factor of importance, a method is developed in the fifth section that analyzes the propagation of the waiting time for a singular perturbation of the headways. The sixth and seventh sections discuss, respectively, the objective function that is related to the operational cost and a perturbation method for minimizing this function. The final section discusses the optimum assignment of buses to the trips.

ORGANIZATIONAL STRUCTURE

THREE FACTORS relevant to the present investigation that relate to the organizational structure are economic structure, personnel structure, and the executive structure. The effect of these on making a timetable and bus schedule are described below.

Economic Structure

From a global aspect the economic structure of the organization under study is mixed because the bus transport that is in the public sector faces competition from electric railway transport that functions independently.

For the particular investigations described, however, this undertaking operates as a monopoly.

Personnel Structure

The number of statisticians employed by the organization are limited. Consequently, the collection of data in the field has to be entrusted to less experienced employees. This imposes a serious restriction on the complexity of the data that can be collected if reliability of the results were not to be affected. The crew for manning the buses are governed by labor laws and as such greatly influence the timetable and bus schedule. There are also several stipulations that have been agreed upon between management and crew, with the conveniences of the crew in view. For example, the crew start their daily work schedule from a depot nearest to their residence and the last trip should terminate nearest to this same depot.

Executive Structure

A timetable, optimum or otherwise, could only be as effective as the accuracy with which it is executed in the field. Consequently, it is necessary to take into consideration the executive structure. The more efficiently the timetable is implemented, the steeper will be the probability density function of punctuality, while the standard deviation will be less. Figure 1 shows the probability density function (not to scale) of the time of arrival or departure of busses with (i) the nonoptimum timetable as the mean value and a given probability density, (ii) the optimum timetable as the mean value and the same probability density, and (iii) the optimum timetable as the mean value and a smaller probability density.

Since the probability density function is centered on the specified time of departure or arrival of buses, an economy would still be effected by optimization in spite of the distribution that is bound to exist when human factors are involved. Punishment or incentive schemes such as fines or reprimands or punctuality rewards will, to some extent, reduce the dispersion and hence the standard deviation. However, the mean value of the distribution, which is the most important factor, is independent of unintentional or unavaoidable delays by the crew in meeting the schedule. The organization under study has a moderate punishment and incentive scheme that appears to be sufficient to keep the dispersion to a reasonable limit.

STATISTICAL CONSIDERATIONS

TWO CLASSES of statistical estimates were needed as inputs to the computer—a priori estimate forming the main input and an a posteriori estimate that

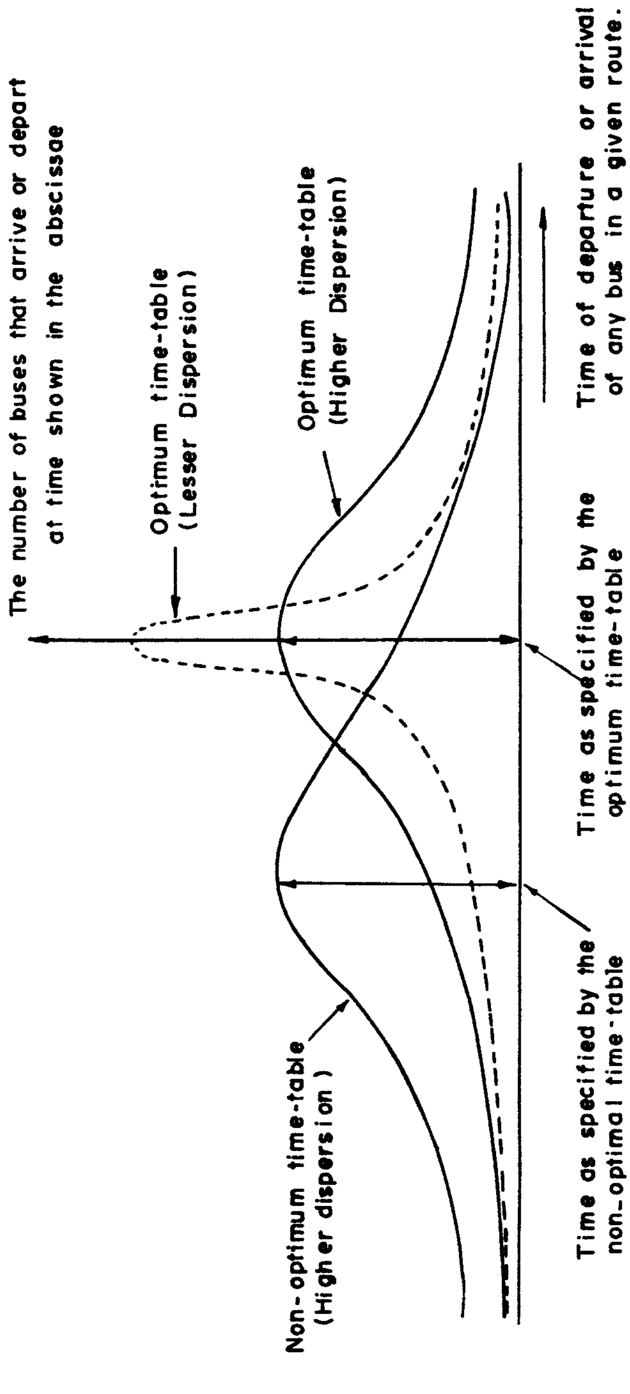

Fig. 1. Probability Density function of punctuality.

adjusts the value of certain parameters in the program with the queue length under consideration.

The a priori statistics relate to three variables on a given bus route, assumed fixed for the rest of the discussion. These are: the number of passengers alighting from the bus, A_{ij}, the number of passengers boarding, B_{ij}, and the time, T_{ij}, at which the bus reached the stop i during the jth trip. For obtaining this data, a person was assigned to each bus operating on the route for a period of four days. The data so obtained at each stop was cumulated at each stage, and the daily history of each bus was maintained in time sequencee.

The a posteriori statistics concern the collection of information about the waiting time of passengers at various stages in the route collected at different times of the day for (i) the execution of the nonoptimal timetable in current use and (ii) the execution of the optimal timetable proposed. For effecting this, a person located at each stage point on the route joins the queue at a particular time and notes down the time at which he reaches the head of the queue. He rejoins the queue immediately thereafter for the next experiment. This procedure is followed over the entire operating time of the day.

The following points will clarify the reasons for adopting the above methods:

1. The variables A_{ij} and B_{ij} at the ith stage for the jth trip are required for computing the vacancies in the bus at each stage for the present headway and for computing new headways.

2. Time T_{ij} is required for the following reason. T_{ij} is based on the present timetable and, hence, the present headway allocation. With this information one can correlate the existing headway with the load at a particular time at a given stop and the number of vacancies, and use this correlation for determining the new headway list so as to minimize the cost function subject to the maximum permissible waiting time.

3. The data are collected from within the bus because of the need to record T_{ij}, accurately, for the history of each bus.

4. Since the passengers in the actual queue have several alternative routes available to them for travel, it may increase the complexity of the data if the queue length for a particular bus is required. For reasons given in the section, "Personnel Structure," such a complexity is not desirable.

5. Stagewise cumulation is adopted because the fare structure is stagewise.

6. A two-day history selected for consistency from out of the four-day history was found to be adequate for obtaining consistent computer output.

7. When considering all computations relative to the present headway list, one can easily show that the list of vacancies itself can be a variable reflecting the waiting time of the passengers. The successive occurrence of nonzero vacancies would indicate a waiting time of less than the duration of one headway. On the contrary, this will not be apparent when several zero vacancies arise successively. This situation necessitated the introduction of the a posteriori statistics.

8. The a posteriori statistics are intended for the following purpose. In the course of the computation the maximum waiting time will be related to a single parameter, an increase or decrease of which, respectively, increases or decreases the maximum waiting time. The value of this parameter can be estimated from a study of the waiting time curve before and after the execution of the optimum timetable. Thus, the a posteriori statistics facilitate a feedback of information for refining the constructed timetable, if necessary. It should be mentioned that the above indirect method was the consequence of an observed fact, viz., the actual waiting-time survey of routes that are not isolated but have a sublength of the route in common with another may be entirely misleading.

REQUIREMENTS AND CONSTRAINTS

THE MANAGEMENT specified a set of requirements that were in the nature of beneficial attributes to the optimization. As against these, management, crew, and passengers impose certain necessary constraints that tend to retard the effect of the optimization. It is natural, therefore, to consider a constrained optimization procedure to arrive at a compromise between the requirements and the constraints that are listed below.

The management specified the following requirements:

1. The fixed costs should be as low as possible for the given number of passengers transported.
2. The ratio between the fare collected from the passengers and the running cost of vehicles should be as high as possible.
3. The crew should be on duty as near the eight-hour work day as possible.
4. The logging of buses at the termini should be minimized.
5. The layover time of the crew should not exceed eight minutes but an upper bound less than this may be specified for a given route depending upon the running time.

The constraints imposed by the management are:

1. The collection of data should not be very expensive.
2. Computational cost should be as low as possible.
3. The timetable should be of such a format that its execution should be as simple as possible.
4. Discontinuities in the transition of headways and running times should be avoided.
5. The direction of the flow of traffic should be taken into consideration. The optimum headway list should be executed either at the passive terminus or the parent terminus depending upon whether the flow is towards the latter or away from it.
6. The fleet are the same type for a given route: among single-decker or single-decker underslung or double decker.
7. The labor laws should be strictly adhered to.
8. The input format should not involve manual preprocessing.

9. Switching of the buses must be confined to a compact area with a common terminus.

The constraints specified by the crew were:

1. A minimum of two minutes should be given as layover at the parent and the passive termini for each trip.
2. The duty must commence and be completed only at the parent terminus that is nearest to the place of residence.
3. The duty should not exceed eight hours per day, which includes 15 minutes for preparation before the crew's first trip and 15 minutes for depositing cash, etc., after the last trip of the day.
4. A minimum of 30 minutes rest before the completion of 5 hours of work.
5. Rest should preferably be given at the parent terminus.
6. Spreadover should not exceed 12 hours.
7. Straight duties with rest not exceeding 45 minutes are preferred.
8. If a two-part duty is inevitable, then two four-hour duties, one in the morning and one in the evening, with a four-hour rest period, should be adopted.
9. The time spent between two trips in a straight duty should be taken as on duty.

The passengers' requirements imposed the following constraints on the optimization:

1. The maximum waiting time should not exceed 20 minutes for buses with a headway less than 20 minutes and not more than one headway otherwise.
2. The buses should be commissioned regularly, i.e., the variation of the headway must be smooth.
3. The mean-value of the cumulative distribution curve of the waiting time should not exceed 10 minutes and the 22.75 per cent value should not exceed 18 minutes for cases in which the average headway is less than 20 minutes.

WAITING-TIME PROPAGATION

An index that influences the optimization to a considerable extent is the maximum waiting time of the passengers. The given headway list when subjected to a singular perturbation alters the waiting-time pattern. In what follows, a relation between the vacancy and the waiting time is given. This is followed by a description of the waiting-time propagation under a perturbation of the headway. Subsequently, a differential equation is derived whose solution gives the law of propagation.

Maximum Waiting Time and Vacancy

A semi-empirical rule called 'The Third-bus Rule' has been in vogue in many transport organizations. The rule is that a passenger who joins the queue simultaneously with the departure of the first bus, should board a bus, on the average, only after missing the second bus, in order to reduce the variable cost. This is, however, only a thumbrule and is not applicable to the present case because the headway is a variable. Thus, assuming that T_{max} is the specified maximum waiting time, H_r is the headway for the rth bus and p_r is the probability of his boarding the rth bus, it must follow that there exists a k such that

$$p_0 = 0; \quad \sum_{r=1}^{k} p_r = 1; \quad \sum_{r=1}^{k} H_r < T_{max}; \quad \sum_{r=1}^{k+1} H_r > T_{max}. \tag{1}$$

This probability distribution can be converted into a time distribution,

$$T = \sum_{t=1}^{k} [p_t \sum_{r=1}^{t} H_r], \tag{2}$$

where T is the expected waiting time. This is equivalently written, after substituting equations (1), as

$$T = H_1 + \sum_{t=2}^{k} [p_t \sum_{r=2}^{t} H_r]. \tag{3}$$

In terms of the nonzero vacancies and headways, it is possible to express equation (3) as

$$T = H_1 + \sum_{t=2}^{k} [v_t(\sum_{r=2}^{t} H_r / \sum_{r=2}^{k} v_r)], \tag{4}$$

where v_r is the vacancy in the rth bus, i.e., the number of people required to fill the rth bus to capacity. For the special case in which $k = 1$, the maximum waiting time is greater than or equal to the headway; for that case it may be impossible to follow the condition that the maximum waiting time should be less than 20 minutes.

The above is not valid when several zero vacancies arise successively. In such a case, the waiting time is taken into consideration by the following method:

Propagation of Waiting Time

A list of headways $\{H_j \mid j = 1, \cdots, N\}$ and a list of time of commissioning $\{T_j \mid j = 0, \cdots, N\}$, which correspond to the list of trips $\{j \mid j = 0, \cdots, N\}$ has been obtained along with a list of cumulative vacancies $\{v_{ij}\}$ for the jth trip at the ith stage from the data of field experiments carried out with the existing nonoptimal timetable. The present waiting time W_{ij} is a function of $\{H_m \mid j > m\}$. If one of the headways H_k is changed, there will be a consequent change in W_{ij} for all $j \geqq k$. For an increment, $\bar{H}$, if $H_k' = H_k + \bar{H}$, then for all $j \geqq k$, $v_{ji}' = v_{ij} - \bar{v}_{ij}$, subject to the inequality, $\bar{v}_{i,j+1} \leqq \bar{v}_{i,j}$. Under this change:

1. When, for $j \geqq k$, v_{ij} is progressively decreasing from a nonzero vacancy to the nearest vacancy that is equal to zero, the waiting time propagated ahead in time should not increase the waiting time at stages where the number of successive zero vacancies exceeds unity.

2. When, for $j \geqq k$, v_{ij} is progressively increasing from a nonzero vacancy to a local maxima, the effect on the waiting-time propagated ahead in time is not important. This is because $\bar{v}_{ij}$ decreases progressively while v_{ij} increases so that none of v_{ij} will be reduced to zero in this range.

3. If the list of zero vacancies is maintained constant, then the waiting time will not exceed that before the change.

4. For a fluctuation in the vacancy list from one day to another, the above observations refer to a datum nonzero vacancy instead of the zero vacancy as the minimum. This datum is obtained from the distribution of v_{ij} taken over all the j and for the requisite repetition of experiments for each i.

5. When v_{kj} falls to the datum level within a short interval of time, there should be no decrease in the magnitude of the vacancies, equal to or less than this datum.

The above factors call for a function deciding the propagation in terms of the magnitude of the vacancy rather than time. Any phenomenon that involves a decay of the effect from a cause can be represented by an exponential decay function.[6] Hence the law of propagation can be described by,

$$\bar{v}(\bar{H}, V) = C_1(\bar{H})1\,\{1 - \exp\,[-C_2(\bar{H})V]\}, \tag{5}$$

where $\bar{v}$ is the change in the vacancy V after the headway H is increased by $\bar{H}$. This law is subject to the conservation assumption.*

THE OBJECTIVE FUNCTION

THE FIELD data supply information about the vacancies, time, distance between stages, and the capacity of the bus for the present timetable. For optimizing the fixed plus variable cost, it is necessary to establish an objective function representing it in terms of the above data. From theoretical and practical considerations, it was found out that an effective objective function is given by,

$$F = \sum_{i=1}^{M-1} \sum_{j=1}^{N} \frac{d_i}{(C - v_{ij})}, \tag{6}$$

when M = number of stages, N = number of trips, d_i = distance between the ith and $(i + 1)$st stages, C = capacity of the bus. The justifications for adopting this function are given below.

1. Since the fares are fixed stagewise, the number of passengers traveling between stages will be more relevant to the cost than those between individual stops. This points towards a minimization of either the inverse of the passengers traveling between the stages or equivalently, a minimization of the

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* It is known that passenger traffic is a conservative phenomenon, i.e., a small change in headways will not reduce the total number of people traveling by a given route over periods of a day.

vacancy between the stages. $(C - v_{ij})$ gives the number of passengers traveling from the ith stage of the route in the jth trip.

2. The term d_i is required, because, everything being equal, the vacancy reduction should be effected more for a greater distance than for a smaller one because when more fuel is expected to be consumed between two stages for the same fare structure, it is more economical to reduce the vacancy, more here than for a lesser distance. If, for example,

$$\frac{d_i}{C - v_{ij}} + \frac{d_{i+1}}{C - v_{i+1,j}} = \text{Constant},$$

then, for $d_i > d_{i+1}$, $v_{ij} < v_{i+1,j}$ or $(C - v_{ij}) > (C - v_{i+1,j})$, as required above.

3. The headways are controlled at the parent terminus. Thus, the objective function should correspond to the entire trip requiring a summation over all the stages from the parent to the passive terminus and vice-versa. Since one is interested in the global optimization over the day, a summation over all the trips is carried out.

4. It is seen readily that the actual cost function F_0 and the above objective function F are related as,

$$F_0(d_i, C, v_{ij}, i, j) = \gamma F(d_i, C, v_{ij}, i, j),$$

where γ is a normalizing constant. The value of v_{ij}, which minimizes F_0, also minimizes F.

5. Though other equivalent functions like $\sum_j \sum_i v_{ij} d_i$ were tried, an extensive experimental investigation showed that the above stated objective function gives favorable results.

PERTURBATIONAL OPTIMIZATION

The objective function of the previous section has to be minimized by perturbing the original headway list. The method for effecting it is given by the following steps:

For the ith stage and the jth trip, the number of people alighting and boarding at the noted time T_{ij} are A_{ij} and B_{ij} respectively, for $i = 1, 2, \cdots, M$ and $j = 1, 2, \cdots, N$. The vacancy v_{ij} is given by

$$v_{ij} = v_{i-1,j} + A_{ij} - B_{ij}; \qquad v_{1j} = C - B_{1j}. \qquad (7)$$

The headway H_j is

$$H_j = T_{1,j+1} - T_{1j}. \qquad (j = 1, 2, \cdots, N - 1) \quad (8)$$

The vacancy index I_j is defined by

$$I_j = \frac{1}{M}\left[\sum_{i=1}^{M} v_{ij}\right] - V_D, \tag{9}$$

where V_D is the datum vacancy.

An upper bound for varying the original headway list is necessary because if a large perturbation is effected, the conservation assumption leading to equation (5) will not be valid. By letting the upper bound correspond to the waiting-time propagation law of equation (5), it can be assured that any perturbation of the headway within this bound will keep the maximum wait-

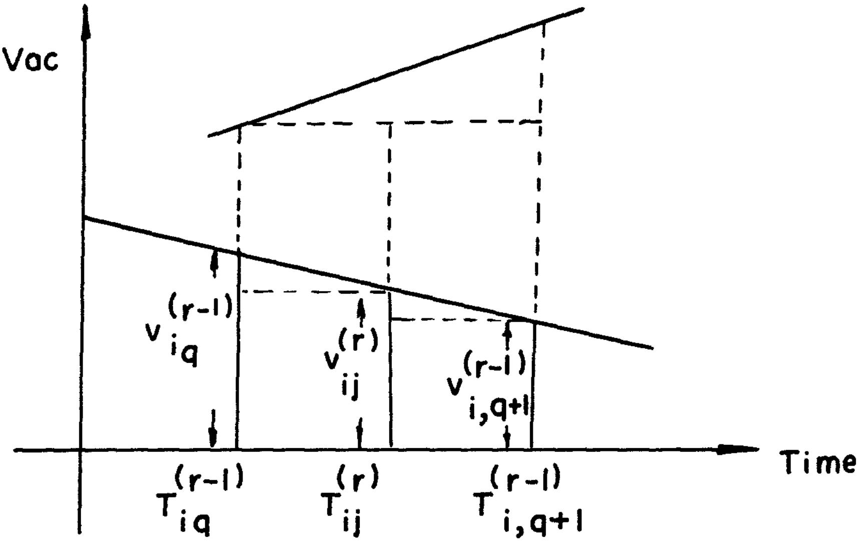

Fig. 2. Interpolation scheme.

ing time equal to or less than that given by the existing timetable (e.g., *see* Fig. 3). The upper bound U_j for the headway H_j is given by

$$U_j = H_j[1 + C_1\{1 - \exp(-C_2 I_{j+1})\}], \tag{10}$$

where C_1 and C_2 are constants of propagation law.

The numerical perturbation is carried out in unit steps upwards of H so that $\bar{H} = 1$. The perturbed headway is given by

$$H_j^{(r)} = H_j^{(r-1)} + 1, \qquad H_j^{(0)} = H_j, \qquad (r = 1, 2, \cdots, g) \tag{11}$$

where $H_j^{(g)} \leq U_j$. The perturbation of the time of arrival of the kth bus at each stage is given by

$$\begin{aligned} T_{i,k}^{(r)} &= T_{i,k}^{(r-1)} + 1, \quad \text{for} \quad k > j, \\ T_{i,k}^{(r)} &= T_{i,k}^{(r-1)}, \quad \text{for} \quad k \leq j. \end{aligned} \tag{12}$$

The vacancy perturbation is realized from considerations based on an interpolation scheme shown in Fig. 2.

If $T_{i,q}^{(r-1)} \leqq T_{i,j}^{(r)} \leqq T_{i,q+1}^{(r-1)}$ and $v_{i,q}^{(r-1)} \geqq v_{i,q+1}^{(r-1)}$, the interpolated vacancy $v_{i-j}^{(r)}$ is obtained from the similar trapeziums by solving the equation

$$(v_{i,q}^{(r-1)} - v_{i,j}^{(r)})(T_{i,q+1}^{(r-1)} - T_{i,j}^{(r)}) = (v_{i,j}^{(r)} - v_{i,q+1}^{(r-1)})(T_{i,j}^{(r)} - T_{i,q}^{(r-1)}). \quad (13)$$

On the other hand, when

$$T_{i,q}^{(r-1)} < T_{i,j}^{(r)} < T_{i,q+1}^{(r-1)}, \qquad \text{and} \qquad v_{i,q}^{(r-1)} < v_{i,q+1}^{(r-1)},$$

it suffices to equate $v_{i,j}^{(r)} = v_{i,q}^{(r-1)}$ because any other value will have a tendency to increase the maximum waiting time in excess of that corresponding to the existing timetable.

Corresponding to the perturbed vacancy, one can calculate the objective function. If this value of objective function shows an improvement over the previous value, the current values of H_j, $T_{i,j}$, and $v_{i,j}$ are accepted. The process is repeated for H_j till either $H_j^{(g)} = U_j$ or no further improvement is achieved on the value of the objective function.

The procedure is carried either from $j = 1$ to $j = NH$, [$NH = N - 1 =$ number of headways] or till the sum of the perturbed headways exceeds the total period of the day for which a route is in operation. In the latter case, NH is replaced by the new number of the headways and N is adjusted accordingly. This completes one main iteration.

The iterations are continued till the global minimum value of the objective function is attained. In actual computation, it was observed that the perturbation procedure is convergent and the rate of convergence is extremely large. For the majority of the routes, only two iterations were sufficient to reach a steady-state value of the objective function and, hence, the optimum headway list.

From the original data of $T_{i,j}$, the running time for each trip is calculated. It is averaged over blocks containing a chosen number of headways. Subsequently, a smooth transition between the successive blocks was effected. From the original values of $v_{i,j}$, the direction of the flow of traffic as well as the transition time were calculated.

SCHEDULING OF BUSES

In the previous section, the optimization was considered with main reference to the variable cost. Reduction in the fixed cost was effected by reducing the number of trips. In this section, it will be shown that this reduction can only be effective when the number of buses employed during the nonpeak hours is minimized. The following procedure was developed to take this aspect into consideration.

The optimum headway list $H(j)$ is averaged selectively along with the running time $R(j)$ by finding out that N_r $(r = 1, 2, \cdots, b)$ for which

$$\sum_{j=k_{r-1}+1}^{k_{r-1}+N_r} H(j) \leqq \frac{1}{N_r} \sum_{j=k_{r-1}+1}^{k_{r-1}+N_r} R(j) + L_{\min}, \tag{14}$$

and

$$\sum_{j=k_{r-1}+1}^{k_{r-1}+N_r+1} H(j) > \frac{1}{N_r + 1} \sum_{j=k_{r-1}+1}^{k_{r-1}+N_r+1} R(j) + L_{\min}, \tag{15}$$

where

$$k_{r-1} = \sum_{i=1}^{r-2} k_i + N_{r-1}. \tag{16}$$

$L_{\min}$ = minimum layover at each terminus. The average headway and running time for the rth block are

$$H_r = \frac{1}{N_r} \sum_{j=k_{r-1}+1}^{k_{r-1}+N_r} H(j), \tag{17}$$

and

$$R_r = \frac{1}{N_r} \sum_{j=k_{r-1}+1}^{k_{r-1}+N_r} R(j). \tag{18}$$

The above averaging facilitates the return of the first bus in the rth block to coincide with the commissioning of the first bus in the $(r + 1)$st block. Since none of the traffic parameters can change abruptly, the average headways are smoothed by increasing or decreasing a subset of the rth block and compensating for the same in the earlier part so as to keep the sum of the headways over any two successive blocks an invariant under the smoothing operation. Let the new list of headways be $H(r, s)$ where s denotes the enumeration of headways within the rth block $(s = 1, 2, \cdots, N_r)$. The departure time $D_T(r, s)$ from the parent terminus for the sth trip in the rth block can be calculated from the recursive rule

$$\begin{aligned} D_T(r, s) &= D_T(r, s - 1) + H(r, s) \qquad \text{for } s \neq 1, \\ D_T(r, 1) &= D_T(r - 1, N) + H(r, 1) \qquad \text{for } s = 1. \end{aligned} \tag{19}$$

The total running time including layover is calculated from

$$R_T(r, s) = D_T(r + 1, s) - D(r, s). \tag{20}$$

The total layover at the two termini are

$$L(r, s) = R_T(r, s) - R_r. \tag{21}$$

$L(r, s)$ can be distributed between two termini as

$$L_1(r, s) = L(r, s)/2 \quad \text{and} \quad L_2(r, s) = L(r, s) - L_1(r, s),$$

where L_1 is the layover at the passive terminus and L_2 is at the parent terminus. The running time without layover is consequently

$$R_0(r, s) = R_T(r, s) - L(r, s). \tag{22}$$

TABLE I

An Example for the Output Format Taken from an Actual Computer Output for a Route of the BEST

TRP NOS	129	130	131	132
BUS NOS	1	2	3	4
PAR DEP TME	22–00	22–08	22–16	22–25
PAR PAS RUN	15	15	16	16
PAS LAY OVR	2	2	2	2
PAS DEP TME	22–17	22–25	22–34	22–43
PAS PAR RUN	15	16	16	16
PAR LAY OVR	2	2	2	2
TRP NOS	133	134	135	136
BUS NOS	1	2	3	4
PAR DEP TME	22–34	22–43	22–52	23–01
PAR PAS RUN	16	16	16	16
PAS LAY OVR	2	2	2	2
PAS DEP TME	22–52	23–01	23–10	23–19
PAS PAR RUN	16	16	16	16
PAR LAY OVR	2	2	2	0
TRP NOS	137	138	139	
BUS NOS	1	2	3	
PAR DEP TME	23–10	23–19	23–28	
PAR PAS RUN	16	16	16	
PAS LAY OVR	2	2	2	
PAS DEP TME	23–28	23–37	23–46	
PAS PAR RUN	16	16	16	
PAR LAY OVR	0	0	0	

(TRP NOS = Trip numbers; BUS NOS = Bus Numbers;
PAR DEP TME = Departure times of the buses from the parent terminus;
PAR PAS RUN = Running times in minutes of the buses from the parent terminus to the passive terminus;
PAS LAY OVR = Layovers in minutes at the passive terminus;
PAS DEP TME = Departure times of the buses from the passive terminus;
PAS PAR RUN = Running times in minutes of the buses from the passive terminus to the parent terminus;
PAR LAY OVR = Layovers in minutes at the parent terminus).

This can be distributed equally or with unit difference between the parent-passive run and the passive-parent run. The departure time at the passive terminus is found out as a consequence.

In the above, it should be noted that suitable corrections for the initial and end blocks should be made. Further, the layover should be taken as zero when a bus in one block is not required in the next. For the same situation, the calculation of $R_T(r, s)$ also follows a slightly different pattern as $D_T(r + 1, s)$ does not exist.

An important operational feature introduced by the above scheduling method, is that for the rth block the buses are commissioned as $\{1, 2, \cdots, N_r\}$ so that the initial sequence of the buses are not altered.

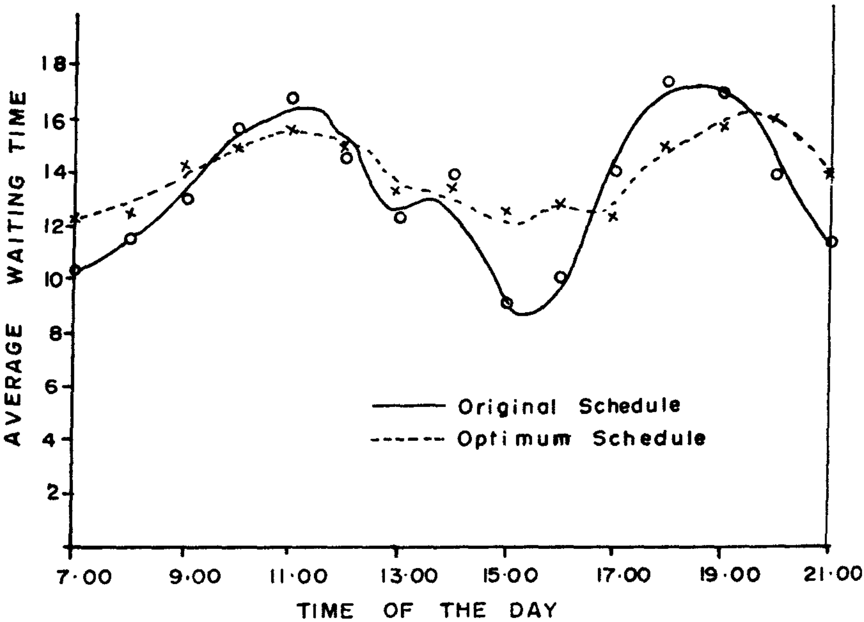

Fig. 3. Variations of average waiting time over a typical day for a test route.

Table I gives an example for the output format taken from an actual computer output for a route of the Bombay Electricity Supply and Transport (BEST) Undertaking.

Fig. 3 gives the average waiting time at various stages on this route before and after the execution of the optimum time table.

CONCLUSION

THE METHOD developed is recommended for large transport organizations in developing countries or medium ones in developed countries that cannot afford to employ a large number of competent statisticians to collect reliable data. The simplicity of the statistical repertory enables the organization to collect reliable data employing only unskilled labor.

A FORTRAN program was written for the method and was executed with the field data furnished by BEST. The optimum timetable and bus schedule resulted as the computer output was made operational on the BEST network. An appreciable reduction in the variable cost was observed. The computational time on CDC 3600 averaged to about 3 minutes per route.

As shown in Fig. 3, which is representative of the results of the field experiments, the effect of the optimum schedule is to straighten out the waiting-time curve. That is, the larger magnitudes of the waiting time of the passengers are reduced and smaller ones are increased.

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